MATH 11: Discussion Week 5

Apr. 2019

Handout and its solution could be found at `https://kimukook.github.io/teaching/math11sp19/`

Binomial distribution Suppose you are practicing shooting arrow. The probability of score is $P(score) = 0.7$.
(a) What is the probability that exactly shooting two scores in five attempts?
Solution:

$$P(X = 2) = \binom{5}{2}(0.7)^2(0.3)^3 = \frac{5!}{2!3!}(0.7)^2(0.3)^3 = 0.1323$$

(b) What is the probability that you have at least scored two times in five attempts?
Solution:

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{5}{0}(0.7)^0(0.3)^5 - \binom{5}{1}(0.7)^1(0.3)^4 \\ &= 0.96922 \end{aligned}$$

Exponential distribution 3. The life of a certain battery is believed to be given by an exponential distribution and the average life is believed to be 3 years.

(a) What is the probability that a randomly selected battery will last longer than 2 years?
Solution:
Let X be the life of battery. Then $X \sim \text{Exp}()$, with $\lambda = \frac{1}{3}$. Because $\mathbb{E}[X] = \frac{1}{\lambda} = 3$. Then

$$P(X > 2) = \int_2^{+\infty} \frac{1}{3} e^{-\frac{x}{3}} dx = e^{-2/3} \approx 0.5134$$

(b) In a pack of 10 batteries, what is the probability at least one of them will last longer than 2 years?
Solution:
Let Y be the number of batteries (out of ten) that last at least 2 years. Then $Y \sim \text{Binomial}(n, p)$, where $n = 10$ and $p = P(X > 2) = e^{-2/3}$. Therefore,

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{10}{0}(e^{-2/3})^0(1 - e^{-2/3})^{10} = 1 - (1 - e^{-2/3})^{10} \approx 0.999256$$

Continuous random variable 1. The random variable X has density

$$f(x) = \begin{cases} kx^2, & \text{if } 0 \leq x \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

(a) Determine the value of k.
Solution:
Since the integral of probability density function should be equal to 1, we have

$$\int_0^3 f(x) dx = \int_0^3 kx^2 dx = \frac{1}{3} kx^3 \Big|_0^3 = 9k = 1$$

Thus, $k = \frac{1}{9}$.

(b) Compute the expected value of X.
Solution:

$$\mathbb{E}[X] = \int_0^3 xf(x)dx = \int_0^3 xkx^2dx = \frac{1}{4}kx^4\Big|_0^3 = \frac{9}{4}$$

2. Suppose some process is modelled by the probability distribution

$$f(x) = \begin{cases} \cos x, & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0, & \text{o.w.} \end{cases}$$

(a) Show that this function meets both requirements of a density function.
Solution:

- Since $\cos x \geq 0$ when $x \in [0, \frac{\pi}{2}]$, $f(x) \geq 0$ for all x.
- $\int_{-\infty}^{+\infty} f(x)dx = \int_0^{\frac{\pi}{2}} \cos xdx = \sin x\Big|_0^{\frac{\pi}{2}}$=1-0=1.

(b) What is $P(X \geq \frac{\pi}{4})$?
Solution:

$$\int_{\frac{\pi}{4}}^{\infty} f(x)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos xdx = \sin x\Big|_{\frac{\pi}{4}}^{\frac{pi}{2}} = 1 - \frac{1}{\sqrt{2}} \approx 0.293$$

(c) What is $P(X = \frac{\pi}{6})$?
Solution:
$P(X = \frac{\pi}{6}) = 0$.